Continuous Optimization from Variational Analysis' viewpoint

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Think about an extension of tangent space to any arbitrary set, not necessarily a manifold (Tangent cone)

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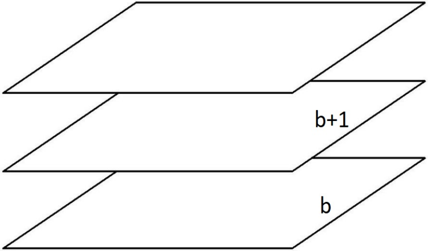

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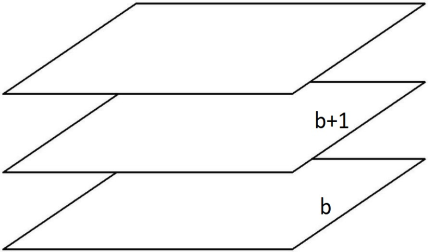

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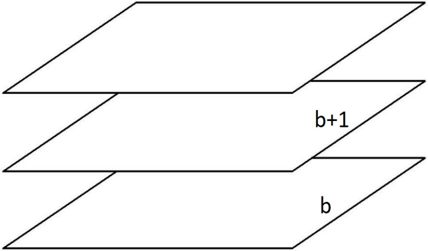
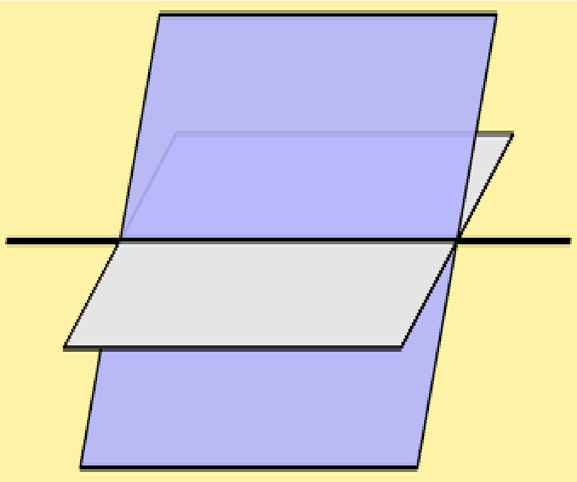


- Consider the set-valued mapping $b \longmapsto \{x|\ Ax = b\}$
- Exercise: Study the continuity of the above set-valued mapping.

Example: How about the set-valued mapping $A \longmapsto \{x|\ Ax = b\}$?



Example: Stability of PDE's and ODE's solution w.r.t coefficients; for example, study the continuity of the following set-valued mapping from $\mathbb{R}$ to $C^2[0,1]$

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Example: Let C be a closed set, $x \in C$, and $f(x) := \operatorname{dist}(x; C)$. Then, what could be

$$\partial f(x) =? \quad \text{and} \quad \partial^2 f(x) =?$$

$$T_C(x) =? \quad \text{and} \quad N_C(x) =?$$

Refrencess

- Rockafellar RT, Wets RJ-B (2006) Variational analysis. Springer, Berlin
- Ioffe AD (2017) Variational analysis of regular mappings: theory and applications. Springer, Cham, Switzerland
- Mordukhovich B.S (2018) Variational Analysis and Applications

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- Assumptions in Numerical Implementation: In addition to the above assumption, we need C to be a nicer set! Like the projection on C has a closed form or $\operatorname{dist}(y; C)$ can be determined with reasonable effort.

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- set $C = \mathbb{R}^k_- \times \{0\}^{m-k}$ (NLP)

$$\begin{aligned} \text{minimize} \quad & \varphi(x) \\ \text{subject to} \quad & f_i(x) \leq 0, \quad i = 1, .., k \\ & f_i(x) = 0, \quad i = k+1, ..., m \end{aligned}$$

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Exercise: For the above C find a closed form $\operatorname{dist}(y; C)$.

An open problem: Is there a closed form of $\operatorname{dist}(y;C)$ where C is a polyhedral convex set.

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A convex SOCP is better than a non-convex nonlinear programming

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \|Ax - b\|_2 \le d^T x + f \end{array}$$

Do not square the constraint, instead do the following:

$$\begin{bmatrix} A \\ d^T \end{bmatrix} x - \begin{bmatrix} b \\ f \end{bmatrix} \in C$$

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Linear SDP is not a LP, but they have a lot in common

$$\begin{array}{ll} \text{minimize} & tr(C^T X) \\ \text{subject to} & tr(A_i^T X) \leq b_i, \quad i = 1, .., k \\ & X \succeq 0 \end{array}$$

Exercise: Find a closed form of $\text{dist}(Y; S^m_+)$.

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Exercise: Show that $LP \subset SOCP \subset SDP$.

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- What can we do then? We can find the candidates of optimality = Necessary optimality conditions.

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- how to fix it? What to do with other Cs? Answer is to rely on Variational Analysis.

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- Exercise: Is there a differentiable extension?(Next Example)

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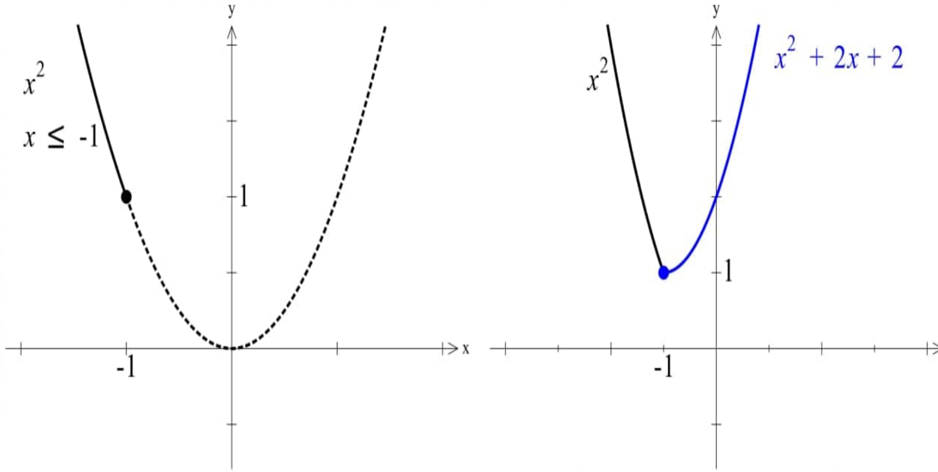
- Look for a nice extension: At least a continuous extension.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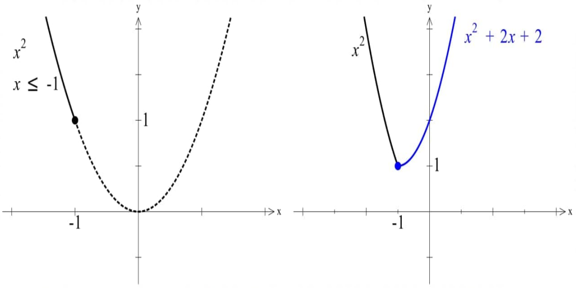

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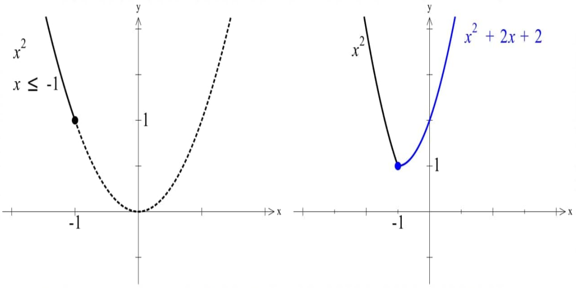

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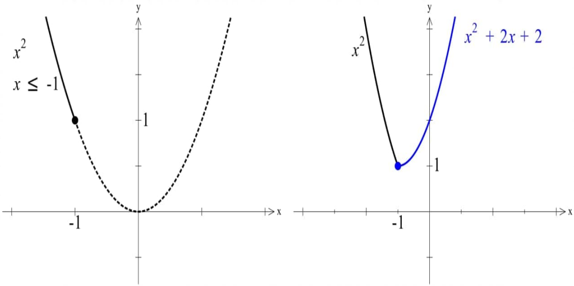

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- Exercise: Show that the new function is given by the following formulas

$$f(x) = x^2 + 2\max\{x + 1, 0\} = x^2 + 2\,\mathrm{dist}(x + 1; (-\infty, 0])$$

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MSCQ Metric subregularity constraint qualification at $\bar{x}$:

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Exact Penalty (Ioffe P 336): Let $MSCQ$ hold at $\bar{x}$. Then, $\bar{x}$ is a local minimizer of P if and only if $\bar{x}$ is a local minimizer of problem

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- In Linear programming, the MSCQ holds for free (Hoffman's Lemma 1951)
- In general Metric regularity implies MSCQ (Mordukhovich Criterion)

$$\ker \nabla F(\bar{x})^* \cap N_C(F(\bar{x})) = \{0\} \implies \text{MSCQ}$$

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- MSCQ/Error bound plays vital role in nonsmooth calculus, stopping criteria, and convergence theory of numerical algorithms

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- Recall: Let $\bar{y} \in C$ (remmeber C is convex). Normal cone to C at y:

$$N_C(y) = \{v \in \mathbb{R}^n \mid \langle v, y-\bar{y}\rangle \leq 0 \text{ for all } y \in \mathbb{R}^m\}.$$

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Example (RW. p340): Let $\bar{x} \in C$ and $f(x) := \mathrm{dist}(x; C)$. Then, $\partial f(\bar{x}) = N_C(\bar{x}) \cap B$. In particular, for $C = \{0\}$ we have $\partial \|.\|(0) = B$

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Example (NLP): Let $C = \mathbb{R}^k_- \times \{0\}^{m-k}$ and $\bar{y} \in C$. Then

$$N_C(\bar{y}) = \{v = (\lambda, \mu) \in \mathbb{R}^k \times \mathbb{R}^{m-k} \mid \lambda \geq 0,\ \langle(\lambda, \mu), \bar{y}\rangle = 0\}$$

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Exercise: If $\bar{x}$ is a local minimizer of f, then $0 \in \partial^{-} f(\bar{x})$.

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Question: How to calculate $\partial^{-} f(x)$ where $f(x) = \mathrm{dist}(F(x); C)$?
Answer: we need a nonsmooth chain rule (next slide).

First-order calculus:

Theorem .
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Let g be convex and f be locally Lipschitz functions. Then, the following calculus rules hold:

- $$\partial^-(g \circ F)(\bar{x}) = \nabla F(\bar{x})^* \partial g(F(\bar{x}))$$
- $$\partial^-(f + g)(\bar{x}) = \partial^- f(\bar{x}) + \partial^- g(\bar{x})$$

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$$\text{minimize}\ \ \varphi(x) \quad \text{over all}\ \ F(x) \in C,$$

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Theorem (First-order optimality condition)

Let MSCQ hold at a (local) minimizer $\bar{x}$. Then

$$\text{there exists } \lambda \in \mathbb{R}^m \text{ with } \begin{cases} 0 = \nabla\phi(\bar{x}) + \nabla F(\bar{x})^*\lambda \\ \lambda \in N_C(F(\bar{x})) \end{cases}$$

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Under MSCQ $\bar{x}$ is a local minimizer of the following problem.

$$\text{minimize} \quad \varphi(x) + \kappa \operatorname{dist}(F(x); C) \quad \text{over all} \quad x \in \mathbb{R}^n,$$

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- $$\begin{aligned} 0 &\in \partial^-\big(\varphi + \kappa \,\mathrm{dist}(F(.); C)\big)(\bar{x}) \\ &= \nabla\varphi(\bar{x}) + \kappa \,\nabla^* F(\bar{x})\, \partial \mathrm{dist}(F(\bar{x}); C) \\ &= \nabla\varphi(\bar{x}) + \nabla^* F(\bar{x}) \,\big[N_C(F(\bar{x})) \cap \kappa B\big] \end{aligned}$$

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- $$\exists\lambda \in N_C(F(\bar{x})) \quad \text{such that} \quad 0 = \nabla\phi(\bar{x}) + \nabla F(\bar{x})^{*}\lambda$$

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$$\text{minimize} \quad \varphi(x) \quad \text{over all} \quad F(x) = 0,$$

$$\exists \lambda \in \mathbb{R}^m \quad \text{such that} \quad \nabla \phi(\bar{x}) = \lambda^T \, \nabla F(\bar{x})$$

Nonlinear Programming:

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- Set $C = \mathbb{R}^k_- \times \{0\}^{m-k}$ (NLP)

$$\begin{aligned} \text{minimize} \quad & \varphi(x) \\ \text{subject to} \quad & f_i(x) \le 0, \quad i = 1, .., k \\ & f_i(x) = 0, \quad i = k+1, ..., m \end{aligned}$$

Then by one of our exercises

$$N_C(F(\bar{x})) = \{v = (\lambda, \mu) \in \mathbb{R}^k \times \mathbb{R}^{m-k} \mid \lambda \ge 0, \ \sum_{i=1}^{k} \lambda_i f_i(\bar{x}) = 0\}$$

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- KKT:

$$\exists(\lambda, \mu) \text{ s.t } \begin{cases} \nabla\varphi(\bar{x}) + \sum_{i=1}^{k} \lambda_i \nabla f_i(\bar{x}) + \sum_{i=K+1}^{m} \mu_i \nabla f_i(\bar{x}) = 0, \\ \lambda_i \ge 0 \ \text{ and } \ \lambda_i f_i(\bar{x}) = 0 \ \text{ for } \ i = 1, \ldots, k \end{cases}$$

Refrencess

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THANKS FOR YOUR ATTENTION